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On the Projectile-Target Duality of the Color Glass Condensate in the Dipole Picture *

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Abstract

Recently Kovner and Lublinsky proposed a set of equations which can be viewed as dual to JIMWLK evolution. We show that these dual equations have a natural dipole-like structure, as conjectured by Kovner and Lublinsky. In the high energy large N_c limit these evolution equations reduce to equations previously derived in the dipole model. We also show that the dual evolution kernel is scheme dependent, although its action on the weight functional describing a high energy state gives a unique result.

Keywords: JIMWLK equation, Balitsky equation, Kovchegov equation, BFKL equation, Dipole Model, Fluctuations, Correlations

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1 Introduction

The Balitsky-JIMWLK equation [1–4] are equations governing the small-x QCD evolution for dense partonic systems. The Balitsky equations are an infinite hierarchy of coupled equations expressing the energy dependence of the scattering of high energy quarks and gluons (represented by Wilson lines in the fundamental and adjoint representations, respectively) on a target. The JIMWLK equation is a functional Fokker-Planck equation [4, 5] for the small-x evolution of the target wavefunction equivalent to the Balitsky equations. The Kovchegov [6] equation is a simplified version of the Balitsky equations where correlations are suppressed, leading to a relatively simple nonlinear equation for the elastic scattering amplitude.

It has recently [7, 8] been realized that the Balitsky-JIMWLK equations miss some essential ingredients in satisfying unitarity constraints in a realistic manner. While these equations accurately handle the recombination of gluons when the gluon occupation number is large they do not properly create the growth of the occupation number starting from a dilute system. For that reason they are accurate, at least in a limited energy domain, when starting with a dense wavefunction, such as that of a big nucleus, but they are not accurate starting from a dilute system such as an elementary dipole.

Iancu and Triantafyllopoulos [7, 8] suggested a new equation which consists of the Balitsky hierarchy along with a stochastic term which, in a dipole language, corresponds to dipole creation or dipole splitting. In Ref. [9] Mueller, Shoshi and Wong cast this equation into an equation for the JIMWLK weight function with the addition to the usual JIMWLK Fokker-Planck term being a fourth order functional derivative. Later on in the paper, this extension to the JIMWLK equation will be referred to as the MSW term. The effect of this stochastic term on the saturation momentum and the scattering matrix at asymptotic rapidities has been worked out in [10, 11]. Finally, in Ref. [12] both the splitting and recombination terms were written in a compact and simple form in the relevant large N_c limit for the high-energy scattering problem. (Parts of these results were anticipated in Ref. [13].)

Kovner and Lublinsky [14, 15] have suggested a general duality between the equations for low-density and high-density systems. In the above mentioned large N_c limit this duality is manifest [12] and the equations for dipole splitting are exactly those given in Ref. [8, 9]. The general form of the duality suggested in Refs. [14, 15] is likely correct and there is an ongoing effort to make this duality more explicit [16].

In this paper, we address the relationship between the splitting term discussed in Ref. [8, 9] and the more general splitting terms given in Ref. [15]. This is partly an

elaboration of discussions previously given in Refs. [17–19] in a closely related context. In particular we emphasize that the splitting term used in Refs. [8, 9] actually defines what is meant by the large-N_c limit in a high-energy scattering problem. We recall that for the high-energy scattering of two dipoles the leading terms in $\alpha_s\, N_c\, Y$ are in fact just the BFKL parts of the evolution. All multi-BFKL evolutions (multi-pomeron terms) are, strictly speaking, higher order in N_c. However, it is convenient to define a high-energy large-N_c limit where one keeps $1/N_c^2$ terms which are enhanced by a factor $\exp[(\alpha_P - 1)Y]$ [17–19] where Y is the rapidity of the scattering and α_P the usual hard pomeron intercept. This is a natural definition since it is exactly when $\alpha_s^2 \exp[(\alpha_P - 1)Y]$ is the order of one that unitarity corrections become important. In this large-N_c limit there is no difference between the dipole splitting used in Ref. [8, 9] and the more general splitting given in Refs. [14, 15].

While from the point of view of an evolved wavefunction the dominance of the "large-N_c" dipole splittings is manifest it is not so clear how this occurs as one evolves the general vertex (dipole splitting). We examine both generally and explicitly the scattering to two projectile dipoles on a target dipole. At low energies the general splitting term and the "large-N_c" splitting term are not the same and the difference between the two contributions is of the same size as either contribution [15]. However, as one evolves to large rapidity the dominant contribution, involving the "large-N_c" splitting term, behaves as $\alpha_s^4 \exp[2(\alpha_P - 1)Y]$ while the difference between the large-N_c term and the general term behaves as $\alpha_s^4 \exp[(\alpha_P - 1)Y]$. This is a remarkable result. It has a counterpart in pomeron splitting language where Braun and Vacca [20] observed that in a particular, and natural, scheme for defining the triple pomeron vertex [21, 22] that vertex dominates over non-triple pomeron splittings. This dominance seems to be exactly the same as what has been described in comparing our large-N_c splitting and the correction terms which evolve less rapidly in rapidity.

We have also found a surprising result. If one writes the evolution of the JIMWLK weight function as in Eq. (5.2), then we find χ to be scheme dependent while $\chi\, W_Y$ is uniquely defined. We have arrived at this result in an explicit way. We take χ to be given by Eq. (5.10), and we find that the terms involving four derivatives with respect to ρ do not agree with the result, Eqs. (2.38) and (2.39), of Ref. [15]. However, we do not believe this to be because one of the calculations is incorrect, but rather because the definitions of χ are different in the two cases, Eq. (2.33) in [15] and Eq. (5.10) here. We believe both definitions of χ are correct and that when a physical quantity, such as the right hand side of Eq. (5.2), is calculated agreement will be reached. We have explicitly evaluated all the terms involving fourth order derivatives of ρ on the right hand side of Eq. (5.2), and we find the generalized dipole picture of Kovner and Lublinsky[15] emerges. This can

be viewed as an explicit check of the more general derivation of evolution of the generalized dipole picture of the weight function W_Y given in Sec. 3. (This result has also recently been found in Ref. [23].)

The outline of the paper is as follows: In Sec. 2 we briefly review JIMWLK evolution and the dual evolution discussed in Refs. [14, 15]. In Sec. 3 we show that the dual evolution follows that of a generalized dipole picture as previously suggested by Kovner and Lublinsky [15]. In Sec. 4 we show that the splitting terms not included in the large-N_c picture evolve much more slowly in energy than those included in the high-energy large-N_c approximation. In Sec. 5 we explicitly evaluate terms through four derivatives in ρ and discuss the scheme dependence of the evolution kernel.

2 The evolution equation of the color glass condensate

2.1 The Balitsky-JIMWLK equation

For a left moving projectile with lightcone time x^- which experiences at the time of the collision the effect of the color fields $\alpha^a(x^-, \boldsymbol{x})$ originated from the target, the Balitsky-JIMWLK equation is

$$\frac{\partial}{\partial Y} W_Y[\alpha] = \frac{1}{2} \int_{\boldsymbol{x},\boldsymbol{y}} \frac{\delta}{\delta \alpha^a(\boldsymbol{x})} \, \eta^{ab}(\boldsymbol{x}, \boldsymbol{y})[\alpha] \, \frac{\delta}{\delta \alpha^b(\boldsymbol{y})} W_Y[\alpha] \ , \tag{2.1}$$

where $Y = \ln 1/x$ is the rapidity of the small-x gluons, $\int_{\boldsymbol{x}} \equiv \int d^2\boldsymbol{x}$ ($\boldsymbol{x}$, $\boldsymbol{y}$ denote transverse coordinates), α^a is the color gauge field radiated by the color sources in the target, and the functional $W_Y[\alpha]$ is the weight function for a given field configuration. The functional derivatives $\delta/\delta\alpha^a(\boldsymbol{x})$ here are meant to be derivatives at the largest x^- or equivalently they stand for

$$\frac{\delta}{\delta \alpha^a(\boldsymbol{x})} \equiv \lim_{x^- \to \infty} \frac{\delta}{\delta \alpha^a(x^-, \boldsymbol{x})} \ . \tag{2.2}$$

The kernel $\eta^{ab}(\boldsymbol{x}, \boldsymbol{y})[\alpha]$ is a non–linear functional of α,

$$\eta^{ab}(\boldsymbol{x}, \boldsymbol{y})[\alpha] = \frac{1}{\pi} \int_{\boldsymbol{z}} \mathcal{K}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \, (1 + \tilde{V}^\dagger_{\boldsymbol{x}} \tilde{V}_{\boldsymbol{y}} - \tilde{V}^\dagger_{\boldsymbol{x}} \tilde{V}_{\boldsymbol{z}} - \tilde{V}^\dagger_{\boldsymbol{z}} \tilde{V}_{\boldsymbol{y}})^{ab} \ , \tag{2.3}$$

as it depends on the Wilson lines in the adjoint representation $\tilde{V}$ and $\tilde{V}^\dagger$,

$$\tilde{V}^\dagger_{\boldsymbol{x}}[\alpha] \ = \ \mathrm{P} \exp \left(\mathrm{i} g \int dx^- T^a \alpha^a(x^-, \boldsymbol{x}) \right) \tag{2.4}$$

with $(T^a)_{bc} = -if^{abc}$, where P denotes the path–ordering along the light-cone coordinate x^- and

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \frac{1}{(2\pi)^2}\frac{(\boldsymbol{x}-\boldsymbol{z})\cdot(\boldsymbol{y}-\boldsymbol{z})}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \; . \tag{2.5}$$

2.2 The dual equation to Balitsky-JIMWLK

Given the JIMWLK equation, the dual equation can be obtained from JIMWLK via the duality transformations [12, 14–16]

$$x^- \Longleftrightarrow x^+ \; , \qquad \frac{\delta}{\delta\alpha^a(x^-,\boldsymbol{x})} \Longleftrightarrow i\rho^a(x^+,\boldsymbol{x}) \; , \qquad \alpha^a(x^-,\boldsymbol{x}) \Longleftrightarrow -i\frac{\delta}{\delta\rho^a(x^+,\boldsymbol{x})} \; . \tag{2.6}$$

This transformation is a consequence of the Lorentz boost invariance and the arbitrariness in the labelling of which of the incoming hadrons is the target and which is the projectile. The resulting dual equation to JIMWLK is

$$\frac{\partial}{\partial Y}W_Y[\rho] = -\frac{1}{2}\int_{\boldsymbol{x},\boldsymbol{y}} \rho^a(\boldsymbol{x})\,\eta^{ab}(\boldsymbol{x},\boldsymbol{y})[-i\,\delta/\delta\rho]\,\rho^b(\boldsymbol{y})W_Y[\rho] \tag{2.7}$$

where $\rho^a(\boldsymbol{x})$ is now the two-dimensional charge density of the target at the largest x^+, so similarly to Eq. (2.2) the following short notation is used

$$\rho^a(\boldsymbol{x}) = \lim_{x^+\to\infty}\rho^a(x^+,\boldsymbol{x}) = \lim_{x^+\to\infty}\int dx^-\,\rho^a(x^+,x^-,\boldsymbol{x}) \; . \tag{2.8}$$

The new $\eta^{ab}[-i\delta/\delta\rho]$ is given by

$$\eta^{ab}(\boldsymbol{x},\boldsymbol{y})[-i\,\delta/\delta\rho] = \frac{1}{\pi}\int_{\boldsymbol{z}} \mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\,(1+\tilde{U}^\dagger_{\boldsymbol{x}}\tilde{U}_{\boldsymbol{y}} - \tilde{U}^\dagger_{\boldsymbol{x}}\tilde{U}_{\boldsymbol{z}} - \tilde{U}^\dagger_{\boldsymbol{z}}\tilde{U}_{\boldsymbol{y}})^{ab} \; , \tag{2.9}$$

with

$$\tilde{U}^\dagger_{\boldsymbol{x}} = \tilde{V}^\dagger_{\boldsymbol{x}}[-i\,\delta/\delta\rho] \;=\; \mathrm{P}\exp\left(g\int dx^+ T^a\delta/\delta\rho^a(x^+,\boldsymbol{x})\right) \; . \tag{2.10}$$

Here P indicates, as usual, an ordering of the integral where terms having larger values of x^+ come further to the left. Whereas in the JIMWLK equation the emphasis is on the functional variable $\alpha(x^-,\boldsymbol{x})$, the field experienced by the projectile, the dual equation has the emphasis on the two-dimensional density, $\rho(x^+,\boldsymbol{x})$, of the target.

3 The color dipole version of the dual equation to JIMWLK

In order to arrive at a color dipole version of Eq. (2.7) upon which all our further discussions will be based, we examine two cases of the W_Y for a single dipole and for two dipoles respectively to all orders in derivatives in ρ. At the end we make a brief remark on the case of an arbitrary number of dipoles.

3.1 The case of a single dipole

We denote a single dipole as

$$W_Y^{(0)}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{N_c}\mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}})\,\delta[\rho] \equiv R(\boldsymbol{x},\boldsymbol{y})\,\delta[\rho]\ . \tag{3.1}$$

$U^\dagger_{\boldsymbol{x}}$ is given by an identical expression as $\tilde{U}^\dagger_{\boldsymbol{x}}$ in Eq. (2.10) but in the fundamental representation of $SU(N_c)$. The superscript (0) in Eq. (3.1) stands for the fact that one begins with an unevolved dipole, with Y-evolution being given by Eq. (3.2). Thus at first order in the evolution

$$\frac{\partial}{\partial Y}W_Y^{(1)}(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{2}\int_{\boldsymbol{u},\boldsymbol{v}} \rho^a(\boldsymbol{u})\,\eta^{ab}(\boldsymbol{u},\boldsymbol{v})[-i\,\delta/\delta\rho]\,\rho^b(\boldsymbol{v})\,W_Y^{(0)}(\boldsymbol{x},\boldsymbol{y})\ . \tag{3.2}$$

The action of $\rho^a(\boldsymbol{v})$ on $W_Y^{(0)}(\boldsymbol{x},\boldsymbol{y})$ is

$$\rho^b(\boldsymbol{v})\,W_Y^{(0)}(\boldsymbol{x},\boldsymbol{y}) = \frac{g}{N_c}\,\mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b)\,(\delta^2(\boldsymbol{v}-\boldsymbol{y}) - \delta^2(\boldsymbol{v}-\boldsymbol{x}))\,\delta[\rho] \tag{3.3}$$

and

$$\begin{aligned}\rho^a(\boldsymbol{u})\rho^b(\boldsymbol{v})\,W_Y^{(0)}(\boldsymbol{x},\boldsymbol{y}) = \frac{g^2}{N_c}\Big(&-\mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a)\delta^2(\boldsymbol{u}-\boldsymbol{x}) + \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^a t^b)\delta^2(\boldsymbol{u}-\boldsymbol{y})\Big)\\ &\times\ (\delta^2(\boldsymbol{v}-\boldsymbol{y}) - \delta^2(\boldsymbol{v}-\boldsymbol{x}))\,\delta[\rho]\ .\end{aligned} \tag{3.4}$$

To use Eq. (3.3) and Eq. (3.4) one has first to bring $\rho^a(\boldsymbol{u})$ to the right of $\eta^{ab}(\boldsymbol{u},\boldsymbol{v})$. The action of $\rho^a(\boldsymbol{u})$ on $(\tilde{U}_{\boldsymbol{v}})^{cb}$ is

$$\rho^a(\boldsymbol{u})(\tilde{U}_{\boldsymbol{v}})^{cb} = (\tilde{U}_{\boldsymbol{v}})^{cb}\,\rho^a(\boldsymbol{u}) + g\,(\tilde{U}_{\boldsymbol{v}} T^a)^{cb}\,\delta^2(\boldsymbol{u}-\boldsymbol{v})\ . \tag{3.5}$$

Using Eq. (3.3) and Eq. (3.4) in Eq. (3.2) one obtains

$$\frac{\partial}{\partial Y}W_Y^{(1)}(\boldsymbol{x},\boldsymbol{y}) = \frac{g^2}{\pi N_c}\int_{\boldsymbol{z}} \mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\,\{1 + \tilde{U}^\dagger_{\boldsymbol{x}}\tilde{U}_{\boldsymbol{y}} - \tilde{U}^\dagger_{\boldsymbol{x}}\tilde{U}_{\boldsymbol{z}} - \tilde{U}^\dagger_{\boldsymbol{z}}\tilde{U}_{\boldsymbol{y}}\}^{ab}$$

$$
\begin{aligned}
&\qquad\qquad \times\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a)\, \delta[\rho] \\
&- \frac{g^2}{2\pi N_c} \int_{\boldsymbol{z}} \mathcal{K}(\boldsymbol{x},\boldsymbol{x},\boldsymbol{z})\, \{2 - \tilde{U}^\dagger_{\boldsymbol{x}} \tilde{U}_{\boldsymbol{z}} - \tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{x}}\}^{ab}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a)\, \delta[\rho] \\
&- \frac{g^2}{2\pi N_c} \int_{\boldsymbol{z}} \mathcal{K}(\boldsymbol{y},\boldsymbol{y},\boldsymbol{z})\, \{2 - \tilde{U}^\dagger_{\boldsymbol{y}} \tilde{U}_{\boldsymbol{z}} - \tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{y}}\}^{ab}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a)\, \delta[\rho] \\
&- \frac{g^2}{2\pi N_c} \int_{\boldsymbol{z}} \Big(\mathcal{K}(\boldsymbol{x},\boldsymbol{x},\boldsymbol{z})\, \{\tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{x}} T^a\}^{ab} - \mathcal{K}(\boldsymbol{y},\boldsymbol{y},\boldsymbol{z})\, \{\tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{y}} T^a\}^{ab} \Big) \\
&\qquad\qquad \times\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b)\, \delta[\rho]\ .
\end{aligned} \tag{3.6}
$$

To proceed further we note that $\tilde{U}_{\boldsymbol{x}}$ can be expressed entirely in terms of $U_{\boldsymbol{x}}$

$$
\{\tilde{U}_{\boldsymbol{x}}\}^{ab} = 2\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} t^a U_{\boldsymbol{x}} t^b) \tag{3.7}
$$

and one can use the Fierz identity

$$
t^a_{ij} t^a_{kl} = \frac{1}{2} \left(\delta_{il}\, \delta_{jk} - \frac{1}{N_c} \delta_{ij}\, \delta_{kl} \right) \tag{3.8}
$$

so the product of $\tilde{U}$'s can be rewritten as

$$
\{\tilde{U}^\dagger_{\boldsymbol{x}} \tilde{U}_{\boldsymbol{y}}\}^{ab} = \{\tilde{U}^\dagger_{\boldsymbol{x}}\}^{ac} \{\tilde{U}_{\boldsymbol{y}}\}^{cb} = 2^2\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} t^c U_{\boldsymbol{x}} t^a)\, \mathrm{tr}(U^\dagger_{\boldsymbol{y}} t^c U_{\boldsymbol{y}} t^b) = 2\, \mathrm{tr}(U^\dagger_{\boldsymbol{y}} U_{\boldsymbol{x}} t^a U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b)\ . \tag{3.9}
$$

Now repeated use of Eq. (3.8) allows us to work out each term in Eq. (3.6). For example

$$
\{\tilde{U}^\dagger_{\boldsymbol{x}} \tilde{U}_{\boldsymbol{y}}\}^{ab}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a) = \frac{1}{2} \left(N_c - \frac{1}{N_c} \right) \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}})\ , \tag{3.10}
$$

$$
\begin{aligned}
\{\tilde{U}^\dagger_{\boldsymbol{x}} \tilde{U}_{\boldsymbol{z}}\}^{ab}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a) &= \{\tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{y}}\}^{ab}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a) \\
&= \frac{1}{2}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}})\, \mathrm{tr}(U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{y}}) - \frac{1}{2N_c}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}})\ ,
\end{aligned} \tag{3.11}
$$

$$
\{\tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{x}}\}^{ab}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b t^a) = \frac{1}{2}\, \mathrm{tr}(U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{x}})\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}} U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}}) - \frac{1}{2N_c}\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}})\ . \tag{3.12}
$$

In the last line of Eq. (3.6) there are terms with

$$
\begin{aligned}
\{\tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{x}} T^a\}^{ab} &= 2\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}} t^a U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{x}} t^c)\, \{T^a\}^{cb} = -2\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}} t^a U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{x}} (t^b t^a - t^a t^b)) \\
&= -\, \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}}) \mathrm{tr}(U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{x}} t^b) + \mathrm{tr}(U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{x}}) \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}} t^b)\ .
\end{aligned} \tag{3.13}
$$

This contracting with the remaining trace factor gives

$$
\{\tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}_{\boldsymbol{x}} T^a\}^{ab} \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b) = -\frac{1}{2} \Big(\mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}}) \mathrm{tr}(U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{y}}) - \mathrm{tr}(U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{x}}) \mathrm{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}} U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}}) \Big)\ . \tag{3.14}
$$

Substituting everything into Eq. (3.6) all terms accompanied by an explicit $1/N_c$ factor such as the second term in Eq. (3.10), Eq. (3.11) and Eq. (3.12) cancel within each $\{\}^{ab}$ and terms with a trace of four U's also cancel leaving

$$
\begin{aligned}
\frac{\partial}{\partial Y} W_Y^{(1)}(\boldsymbol{x},\boldsymbol{y}) = & -\frac{g^2}{2\pi N_c}\int_{\boldsymbol{z}} (-2\mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) + \mathcal{K}(\boldsymbol{x},\boldsymbol{x},\boldsymbol{z}) + \mathcal{K}(\boldsymbol{y},\boldsymbol{y},\boldsymbol{z})) \\
& \times \Big\{ N_c \operatorname{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}}) - \operatorname{tr}(U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{y}}) \operatorname{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}}) \Big\}\, \delta[\rho] \\
= & -\frac{g^2 N_c}{2\pi}\int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \left\{ R(\boldsymbol{x},\boldsymbol{y}) - R(\boldsymbol{z},\boldsymbol{y}) R(\boldsymbol{x},\boldsymbol{z}) \right\} \delta[\rho]
\end{aligned}
\tag{3.15}
$$

where

$$
\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \frac{1}{(2\pi)^2} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \tag{3.16}
$$

is the kernel of the dipole model. Eq. (3.15) is valid for any values of N_c and is the equation reminescent of the dipole form of the Balitsky-Kovchegov equation that most of the following discussions will be based.

3.2 The case of two dipoles and beyond

Now starting with two dipoles, that part of W_Y is

$$
W_Y^{(0)}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}) = \frac{1}{N_c}\operatorname{tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{w}})\, \frac{1}{N_c}\operatorname{tr}(U^\dagger_{\boldsymbol{w}} U_{\boldsymbol{y}})\, \delta[\rho] \equiv R(\boldsymbol{x},\boldsymbol{w})\, R(\boldsymbol{w},\boldsymbol{y})\, \delta[\rho]\,. \tag{3.17}
$$

Once again we evolve this by using Eq. (2.7) or Eq. (3.2) and eliminating first the ρ's by acting them on $W_Y^{(0)}$. As before there are two possibilities. The first possibility is only one ρ acts on $W_Y^{(0)}$

$$
\rho^b(\boldsymbol{v}) W_Y^{(0)}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}) = R(\boldsymbol{x},\boldsymbol{w}) \left(\rho^b(\boldsymbol{v}) W_Y^{(0)}(\boldsymbol{w},\boldsymbol{y})\right) + R(\boldsymbol{w},\boldsymbol{y}) \left(\rho^b(\boldsymbol{v}) W_Y^{(0)}(\boldsymbol{x},\boldsymbol{w})\right) \tag{3.18}
$$

while the other on η^{ab}, where the $(\rho\, W_Y^{(0)})$'s on the right hand side are given by Eq. (3.3), and the second is both ρ's act on $W_Y^{(0)}$

$$
\begin{aligned}
\rho^a(\boldsymbol{u})\rho^b(\boldsymbol{v}) W_Y^{(0)}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}) = \; & R(\boldsymbol{x},\boldsymbol{w}) \left(\rho^a(\boldsymbol{u})\rho^b(\boldsymbol{v}) W_Y^{(0)}(\boldsymbol{w},\boldsymbol{y})\right) \\
& + R(\boldsymbol{w},\boldsymbol{y}) \left(\rho^a(\boldsymbol{u})\rho^b(\boldsymbol{v}) W_Y^{(0)}(\boldsymbol{x},\boldsymbol{w})\right) \\
& + \left(\rho^a(\boldsymbol{u}) R(\boldsymbol{x},\boldsymbol{w})\right)\left(\rho^b(\boldsymbol{v}) R(\boldsymbol{w},\boldsymbol{y})\right) \delta[\rho] \\
& + \left(\rho^b(\boldsymbol{v}) R(\boldsymbol{x},\boldsymbol{w})\right)\left(\rho^a(\boldsymbol{u}) R(\boldsymbol{w},\boldsymbol{y})\right) \delta[\rho]\,.
\end{aligned}
\tag{3.19}
$$

The two $(\rho\,\rho\,W_Y^{(0)})$'s are given by Eq. (3.4) and the $(\rho\,R)$'s are essentially the same as $(\rho\,W_Y^{(0)})$ in Eq. (3.3) but without the explicit $\delta[\rho]$ factor. These are of course related by

$$\Big(\rho\,R(\boldsymbol{x},\boldsymbol{y})\Big)\,\delta[\rho] = \Big(\rho\,W_Y^{(0)}(\boldsymbol{x},\boldsymbol{y})\Big)\,. \tag{3.20}$$

Based on what we learned in Sec. 3.1 and a careful examination of the form of Eq. (3.18) and Eq. (3.19), it is easy to see that

$$\begin{aligned}\frac{\partial}{\partial Y}\,W_Y^{(1)}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}) = {}& R(\boldsymbol{w},\boldsymbol{y})\,\frac{\partial}{\partial Y}\,W_Y^{(1)}(\boldsymbol{x},\boldsymbol{w}) + R(\boldsymbol{x},\boldsymbol{w})\,\frac{\partial}{\partial Y}\,W_Y^{(1)}(\boldsymbol{w},\boldsymbol{y})\\ &-\frac{1}{2}\int_{\boldsymbol{u},\boldsymbol{v}}\eta^{ab}(\boldsymbol{u},\boldsymbol{v})[-i\delta/\delta\rho]\Big(\rho^a(\boldsymbol{u})R(\boldsymbol{x},\boldsymbol{w})\Big)\Big(\rho^b(\boldsymbol{v})R(\boldsymbol{w},\boldsymbol{y})\Big)\,\delta[\rho]\\ &-\frac{1}{2}\int_{\boldsymbol{u},\boldsymbol{v}}\eta^{ab}(\boldsymbol{u},\boldsymbol{v})[-i\delta/\delta\rho]\Big(\rho^b(\boldsymbol{v})R(\boldsymbol{x},\boldsymbol{w})\Big)\Big(\rho^a(\boldsymbol{u})R(\boldsymbol{w},\boldsymbol{y})\Big)\,\delta[\rho]\,.\end{aligned} \tag{3.21}$$

The expressions in the last two lines can be worked out. The first of these is

$$\begin{aligned}&-\frac{1}{2}\int_{\boldsymbol{u},\boldsymbol{v}}\eta^{ab}(\boldsymbol{u},\boldsymbol{v})[-i\delta/\delta\rho]\Big(\rho^a(\boldsymbol{u})R(\boldsymbol{x},\boldsymbol{w})\Big)\Big(\rho^b(\boldsymbol{v})R(\boldsymbol{w},\boldsymbol{y})\Big)\,\delta[\rho]\\ &= -\frac{g^2}{2\pi N_c^2}\Big\{\int_z \mathcal{K}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})\{1+\tilde U^\dagger_{\boldsymbol{w}}\tilde U_{\boldsymbol{y}}-\tilde U^\dagger_{\boldsymbol{w}}\tilde U_{\boldsymbol{z}}-\tilde U^\dagger_{\boldsymbol{z}}\tilde U_{\boldsymbol{y}}\}^{ab}\\ &\qquad+\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{z})\{1+\tilde U^\dagger_{\boldsymbol{x}}\tilde U_{\boldsymbol{w}}-\tilde U^\dagger_{\boldsymbol{x}}\tilde U_{\boldsymbol{z}}-\tilde U^\dagger_{\boldsymbol{z}}\tilde U_{\boldsymbol{w}}\}^{ab}\\ &\qquad-\mathcal{K}(\boldsymbol{w},\boldsymbol{w},\boldsymbol{z})\{2-\tilde U^\dagger_{\boldsymbol{w}}\tilde U_{\boldsymbol{z}}-\tilde U^\dagger_{\boldsymbol{z}}\tilde U_{\boldsymbol{w}}\}^{ab}\\ &\qquad-\mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\{1+\tilde U^\dagger_{\boldsymbol{x}}\tilde U_{\boldsymbol{y}}-\tilde U^\dagger_{\boldsymbol{x}}\tilde U_{\boldsymbol{z}}-\tilde U^\dagger_{\boldsymbol{z}}\tilde U_{\boldsymbol{y}}\}^{ab}\Big\}\\ &\qquad\times\,\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}t^a)\,\mathrm{tr}(U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}t^b)\,\delta[\rho]\,.\end{aligned} \tag{3.22}$$

Again with the use of the Fierz identity Eq. (3.8), one can work out the terms. For example

$$\{\tilde U^\dagger_{\boldsymbol{w}}\tilde U_{\boldsymbol{y}}\}^{ab}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}t^a)\mathrm{tr}(U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}t^b) = \frac{1}{2}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{y}}) - \frac{1}{2N_c}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}})\mathrm{tr}(U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}})\,, \tag{3.23}$$

$$\begin{aligned}\{\tilde U^\dagger_{\boldsymbol{w}}\tilde U_{\boldsymbol{z}}\}^{ab}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}t^a)\mathrm{tr}(U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}t^b) = {}&\frac{1}{2}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{z}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{w}})\\ &-\frac{1}{2N_c}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}})\mathrm{tr}(U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}})\,,\end{aligned} \tag{3.24}$$

$$\begin{aligned}\{\tilde U^\dagger_{\boldsymbol{z}}\tilde U_{\boldsymbol{w}}\}^{ab}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}t^a)\mathrm{tr}(U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}t^b) = {}&\frac{1}{2}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{z}})\\ &-\frac{1}{2N_c}\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}})\mathrm{tr}(U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}})\,.\end{aligned} \tag{3.25}$$

In fact if one ignores the $\mathcal{K}$ factors, all terms in Eq. (3.22) with $\{\tilde{U}^\dagger \tilde{U}_{\boldsymbol{z}}\}^{ab}$ give the same expression on the right hand side as Eq. (3.24), all terms with $\{\tilde{U}^\dagger_{\boldsymbol{z}} \tilde{U}\}^{ab}$ give Eq. (3.25) and lastly all terms without either $\tilde{U}_{\boldsymbol{z}}$ or $\tilde{U}^\dagger_{\boldsymbol{z}}$ give Eq. (3.23). Substituting all these back into Eq. (3.22), terms with a $1/N_c$ factor cancel among themselves within each $\{\}^{ab}$ and one is left with

$$
\begin{aligned}
& -\frac{1}{2}\int_{\boldsymbol{u},\boldsymbol{v}} \eta^{ab}(\boldsymbol{u},\boldsymbol{v})[-i\delta/\delta\rho]\Big(\rho^a(\boldsymbol{u})R(\boldsymbol{x},\boldsymbol{w})\Big)\Big(\rho^b(\boldsymbol{v})R(\boldsymbol{w},\boldsymbol{y})\Big)\,\delta[\rho] \\
&= -\frac{g^2}{4\pi N_c^2}\int_z \Big(\mathcal{K}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})+\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{z})-\mathcal{K}(\boldsymbol{w},\boldsymbol{w},\boldsymbol{z})-\mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\Big) \\
&\qquad\qquad \times\Big(2\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{y}})-\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{z}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{w}})-\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{z}})\Big)\,\delta[\rho]
\end{aligned}
\tag{3.26}
$$

Incidentally swapping ρ^a and ρ^b in the above equation gives exactly the same result so the last two lines of Eq. (3.21) are identical. Finally Eq. (3.21) becomes

$$
\begin{aligned}
&\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}) \\
&= R(\boldsymbol{w},\boldsymbol{y})\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{x},\boldsymbol{w})+R(\boldsymbol{x},\boldsymbol{w})\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{w},\boldsymbol{y}) \\
&\quad -\frac{g^2}{2\pi N_c^2}\int_z \Big(\mathcal{K}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})+\mathcal{K}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{z})-\mathcal{K}(\boldsymbol{w},\boldsymbol{w},\boldsymbol{z})-\mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\Big) \\
&\qquad \times\Big(2\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{y}})-\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{z}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{w}})-\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{z}})\Big)\,\delta[\rho] \\
&= R(\boldsymbol{w},\boldsymbol{y})\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{x},\boldsymbol{w})+R(\boldsymbol{x},\boldsymbol{w})\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{w},\boldsymbol{y}) \\
&\quad -\frac{g^2}{2\pi N_c}\int_z \Big(\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})-\mathcal{M}(\boldsymbol{w},\boldsymbol{y},\boldsymbol{z})-\mathcal{M}(\boldsymbol{x},\boldsymbol{w},\boldsymbol{z})\Big) \\
&\qquad \times\Big(2R(\boldsymbol{x},\boldsymbol{y})-\frac{1}{N_c}[\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{z}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{w}})+\mathrm{tr}(U^\dagger_{\boldsymbol{x}}U_{\boldsymbol{w}}U^\dagger_{\boldsymbol{z}}U_{\boldsymbol{y}}U^\dagger_{\boldsymbol{w}}U_{\boldsymbol{z}})]\Big)\,\delta[\rho]
\end{aligned}
\tag{3.27}
$$

Apart from the gauge field A_μ's in the U's and $U^\dagger$'s have been replaced by $\delta/\delta\rho$ here, the presence of $\delta[\rho]$ and a prefactor, this last expression is the same equation as Eq. (121) of the first paper in Ref. [1]. Using Eq. (3.15) one can see that the terms in the second line is down by a factor of $1/N_c^2$ in comparison to those in the first line. Therefore in the large N_c limit, the evolution equation for two dipoles simplifies to a "product rule" like form

$$
\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{x},\boldsymbol{w},\boldsymbol{y}) \simeq R(\boldsymbol{w},\boldsymbol{y})\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{x},\boldsymbol{w})+R(\boldsymbol{x},\boldsymbol{w})\frac{\partial}{\partial Y}W^{(1)}_Y(\boldsymbol{w},\boldsymbol{y})\,. \tag{3.28}
$$

Because there are only two ρ's in Eq. (2.7) and they can act on a maximum of two different R's, this product rule holds for the evolution of any arbitrary number of dipoles in the large N_c limit. One can conclude that in this limit each dipole evolves independently of each other.

4 Dipole model versus dual evolution in the low gluon density region

In this section we consider the scattering of two elementary dipoles off an evolved target. The latter is considered to be dilute (low gluon density) so that gluon mergings can be neglected. We construct the wavefunction of an evolved target starting with an elementary dipole by using the dipole model [24, 25] and the dual evolution equation [14–16]. With the evolution operator recently worked out by Mueller, Shoshi and Wong (MSW) [9] to extend the JIMWLK equation to the low gluon density region, we obtain a scattering amplitude which in the dipole model is dominated by the scattering of the two projectile dipoles off *two different* dipoles in the evolved target. Kovner and Lublinsky [15] have shown that the dual evolution equation gives additional terms, missed by the MSW evolution operator, which allow for the two projectile dipoles also to scatter off *a single* dipole in the evolved target. In this section we show that the multiple interaction of the target dipole is suppressed as compared to the single interaction of different target dipoles at high rapidities. The two contributions are comparable only at very low rapidities where the target contains only few gluons. At high rapidities with a large number of gluons in the target the effective operator of Ref. [9] gives correctly the leading contribution to the scattering amplitude.

4.1 Low gluon density evolution in the dipole model

Let us consider the evolution of an unevolved single dipole in the weak field limit

$$W^{(0)}_{Y\,(2)}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4N_c}(\delta^b_{\boldsymbol{x}} - \delta^b_{\boldsymbol{y}})^2\delta[\rho] \equiv R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{y})\,\delta[\rho]\ . \tag{4.1}$$

Here the subscript (2) stands for the second order derivative in ρ and the superscript (0) denotes the unevolved state of the dipole that one begins with. The $\delta^b_{\boldsymbol{x}}$ here is given by

$$\delta^b_{\boldsymbol{x}} = g\int_{-\infty}^{\infty} dx^+ \frac{\delta}{\delta\rho^b(x^+,\boldsymbol{x})}\,. \tag{4.2}$$

The first step in the evolution in the extended JIMWLK equation as given by the dipole model reads [9]

$$\frac{\partial}{\partial Y}\, W_Y^{(1)} \;= (\chi_2 + \chi_4^{\text{MSW}})\; W^{(0)}_{Y\,(2)}\;, \tag{4.3}$$

where the operator χ_2 gives the BFKL evolution and χ_4^{MSW} splits one scattering dipole into two scattering dipoles at every evolution step. In the JIMWLK language, in terms of fields, these have been calculated in Ref. [9]. Using the relation between the fields and charge densities $-\nabla_\perp^2\,\alpha^a(x^+,\boldsymbol{x}) = \rho^a(x^+,\boldsymbol{x})$, they can be expressed in terms of charge densities

$$\chi_2 = \frac{1}{2g^2N_c}\,\frac{g^2N_c}{2\pi}\int_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}}\mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\;\left[(\delta^c_{\boldsymbol{u}}-\delta^c_{\boldsymbol{z}})\,(\delta^d_{\boldsymbol{z}}-\delta^d_{\boldsymbol{v}})\right]\left(T^cT^d\right)_{ab}\;\rho^a(\boldsymbol{u})\rho^b(\boldsymbol{v}) \tag{4.4}$$

and

$$\chi_4^{\text{MSW}} = -\frac{1}{16g^2N_c^3}\,\frac{g^2N_c}{2\pi}\int_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{z}}\mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\;\left[(\delta^c_{\boldsymbol{u}}-\delta^c_{\boldsymbol{z}})^2\right]\,\left[(\delta^d_{\boldsymbol{z}}-\delta^d_{\boldsymbol{v}})^2\right]\rho^a(\boldsymbol{u})\rho^a(\boldsymbol{v})\,. \tag{4.5}$$

Acting with these operators on the $W^{(0)}_{Y\,(2)}(\boldsymbol{x},\boldsymbol{y})$ in Eq. (4.3), one obtains

$$\begin{aligned}\frac{\partial}{\partial Y}\, W_Y^{(1)}(\boldsymbol{x},\boldsymbol{y}) \;&= \frac{g^2N_c}{2\pi}\int_{\boldsymbol{z}}\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \qquad\qquad (4.6)\\ &\times\Big[R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{z}) + R^{(0)}_{(2)}(\boldsymbol{z},\boldsymbol{y}) - R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{y}) + R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{z})R^{(0)}_{(2)}(\boldsymbol{z},\boldsymbol{y})\Big]\,\delta[\rho]\;,\end{aligned}$$

where the first three terms on the right-hand side (r.h.s.) represent the BFKL evolution of the original dipole $(\boldsymbol{x},\,\boldsymbol{y})$ while the fourth term describes the splitting of the original dipole into two new dipoles $(\boldsymbol{x},\,\boldsymbol{z})$ and $(\boldsymbol{z},\,\boldsymbol{y})$ which simultaneously scatter off the projectile. This first step in the evolution of an unevolved single dipole is shown in Fig. 1.

Let us now consider the second step of the evolution. A graphical representation of the result is shown in Fig. 2. This is simply obtained by evolving one step further the graphs on the r.h.s. of Fig. 1 according to the dipole model in the extended evolution, i.e., by acting on them with χ_2 and χ_4^{MSW} as in the first step. In Fig. 2 the first five lines come from the BFKL evolution of the four graphs in Fig. 1, $\chi_2\,(W^{(1)}_{Y\,(2)} + W^{(1),\text{MSW}}_{Y\,(4)})$ with

$$W^{(1),\text{MSW}}_{Y\,(4)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{z})R^{(0)}_{(2)}(\boldsymbol{z},\boldsymbol{y})\;\delta[\rho]\;. \tag{4.7}$$

The last line results by acting with χ_4^{MSW} on the first three graphs in Fig. 1, $\chi_4^{\text{MSW}}W^{(1)}_{Y\,(2)}$. The subscript $[\ ]_4$ in Fig. 2 and Eq. (4.8) indicates that we have

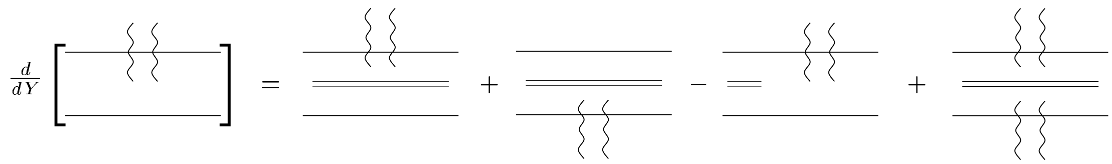

Figure 1: One step of evolution of an unevolved single dipole in the dipole model: The first three graphs on the r.h.s. represent the BFKL evolution of the original dipole and the last graph the splitting of the original dipole into two new dipoles which simultaneously scatter off the projectile dipoles.

dropped in the wavefunction terms having more than four orders in ρ-derivatives, like $\chi_4^{\rm MSW} W_{Y\,(4)}^{(0),\rm MSW}$, since for our purpose it is enough to consider the scattering of the evolved target on only two projectile dipoles.

In formula the second step of the evolution is

$$\frac{\partial}{\partial Y}\left[W_Y^{(2)}\right]_4 = \chi_2\, W_{Y\,(2)}^{(1)} + \chi_2\, W_{Y\,(4)}^{(1),\rm MSW} + \chi_4^{\rm MSW}\, W_{Y\,(2)}^{(1)}\ . \tag{4.8}$$

It is easy to check that this evolution equation holds for any step of the evolution. Summing over all dipoles numbers (superscript), the evolved wavefunction which contains up to four ρ-derivatives reads

$$\frac{\partial}{\partial Y}\left[W_Y\right]_4 = \frac{d}{dY} W_{Y\,(2)} + \frac{\partial}{\partial Y} W_{Y\,(4)}^{\rm MSW} \tag{4.9}$$

with

$$\frac{\partial}{\partial Y} W_{Y\,(2)} \;=\; \chi_2\; W_{Y\,(2)}\ , \tag{4.10}$$

$$\frac{\partial}{\partial Y} W_{Y\,(4)}^{\rm MSW} \;=\; \chi_2\; W_{Y\,(4)}^{\rm MSW} + \chi_4^{\rm MSW}\;\; W_{Y\,(2)}\ . \tag{4.11}$$

In Eq. (4.11) $\chi_4^{\rm MSW}\ W_{Y\,(2)}$ converts one into two simultaneously interacting target dipoles at any step of evolution. Both dipoles evolve in the subsequent steps of the evolution according to the BFKL equation, given by $\chi_2\ W_{Y\,(4)}^{\rm MSW}$, before they scatter on the two projectile dipoles. If the two dipoles are created in the first step of the evolution (fourth graph on the rhs of Fig. 1), then the subsequent BFKL evolution of both dipoles gives for the rapidity dependence of the wavefunction

$$W_{Y\,(4)}^{\rm MSW} \;\sim\; e^{2\,(\alpha_P-1)\,Y}\ W_{Y=0\,(4)}^{\rm MSW} \tag{4.12}$$

with $\alpha_P - 1 = (4\alpha_s N_c \ln 2)/\pi$. When considering the scattering of the evolved target dipole off the two projectile dipoles, (4.12) gives for the T-matrix

$$T^{\,2\,\rm Pom}(Y) \;\sim\; \alpha_s^4\; e^{2\,(\alpha_P-1)\,Y}\ . \tag{4.13}$$

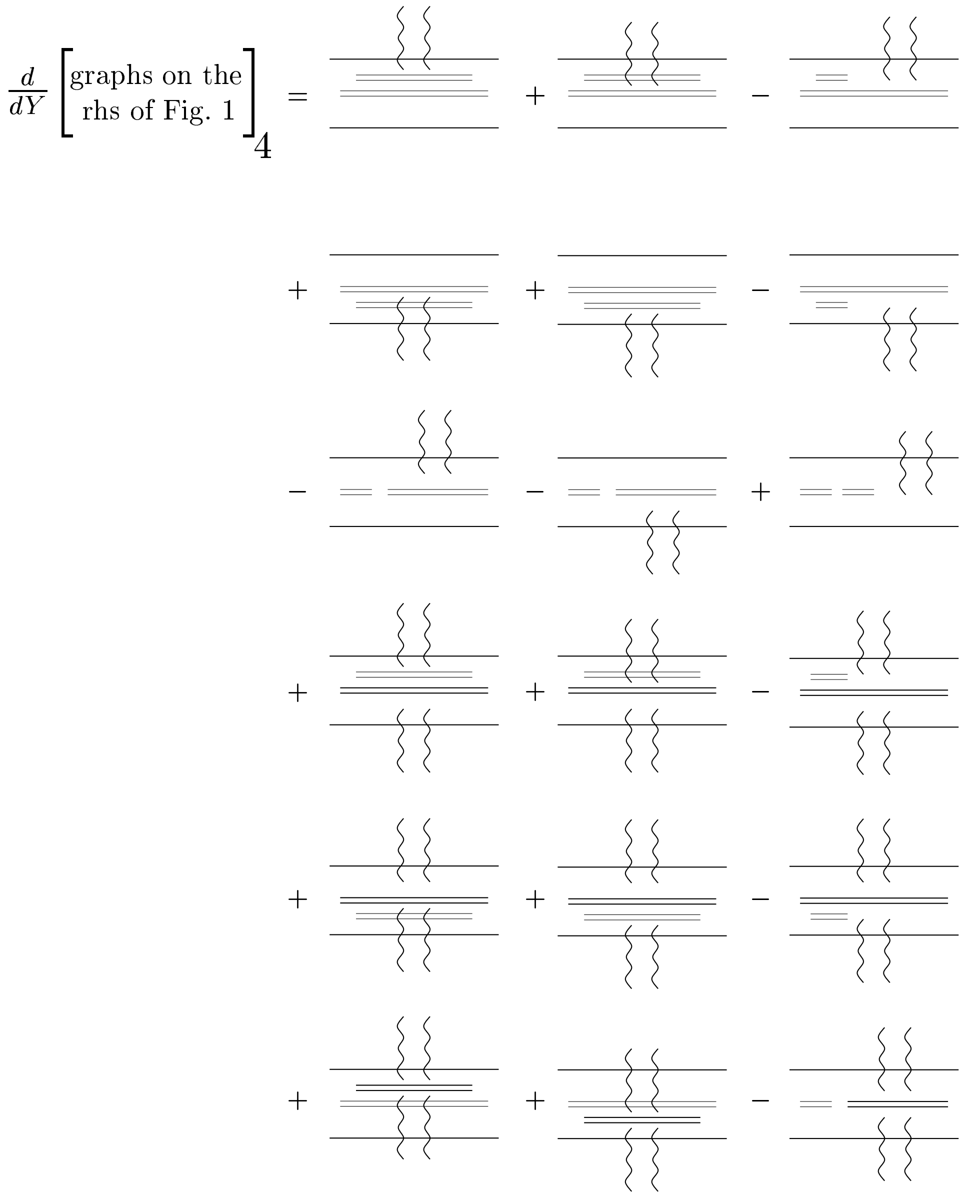

Figure 2: One step of evolution of the graphs on the r.h.s. of Fig. 1 in the dipole model. More than four gluon interactions with the projectile dipoles are excluded.

In the Pomeron language the interaction of the evolved target dipole with the two projectile dipoles occurs via the double Pomeron exchange as shown in Fig. 3(a). However, if the two target dipoles are generated after several previous steps of BFKL evolution of a single dipole (last line in Fig. 2), then the rapidity depen-

dence is partly that of a single Pomeron and partly that of a double Pomeron exchange, $e^{(\alpha_P-1)\,y}e^{2\,(\alpha_P-1)\,(Y-y)}$, which is suppressed as compared with the pure double Pomeron exchange. The later case is shown in Fig. 3(b).

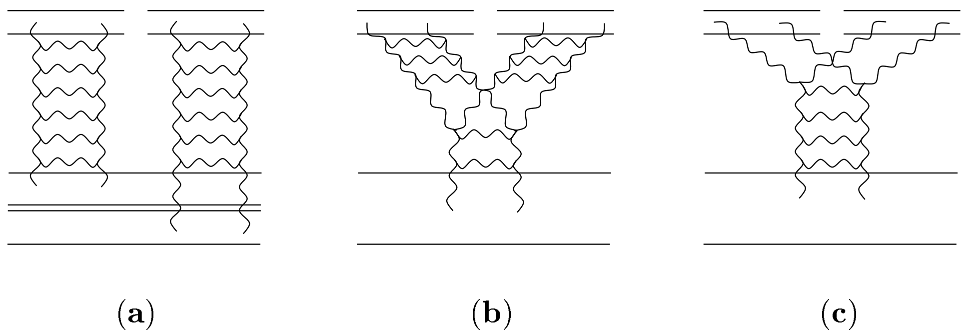

Figure 3: An evolved target scattering off two elementary dipoles in the Pomeron language: (a) double Pomeron interaction, (b) a partly evolved Pomeron splitts into two Pomerons which after some evolution scatter off the two dipoles, (c) a fully evolved Pomeron scatters at the end of its evolution off the two dipoles.

4.2 Low gluon density evolution in the dual equation

The first step of evolution of an unevolved single dipole as given by the dual equation Eq. (2.7) up to the fourth order in ρ-derivatives (see Eq. (3.2) and Eq. (3.15)) is

$$\begin{aligned}\frac{\partial}{\partial Y}\Big[W_Y^{(1)\,\mathrm{dual}}\Big]_4 &= \frac{\partial}{\partial Y}\Big[W^{(1)}_{Y\,(2)}+W^{(1)}_{Y\,(3)}+W^{(1)}_{Y\,(4)}\Big]_4 \\ &= \frac{g^2N_c}{2\pi}\int_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \\ &\quad\times\Big[\;R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{z})+R^{(0)}_{(2)}(\boldsymbol{z},\boldsymbol{y})-R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{y}) \\ &\qquad+R^{(0)}_{(3)}(\boldsymbol{x},\boldsymbol{z})+R^{(0)}_{(3)}(\boldsymbol{z},\boldsymbol{y})-R^{(0)}_{(3)}(\boldsymbol{x},\boldsymbol{y}) \\ &\qquad+R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{z})R^{(0)}_{(2)}(\boldsymbol{z},\boldsymbol{y}) \\ &\qquad+R^{(0)}_{(4)}(\boldsymbol{x},\boldsymbol{z})+R^{(0)}_{(4)}(\boldsymbol{z},\boldsymbol{y})-R^{(0)}_{(4)}(\boldsymbol{x},\boldsymbol{y})\Big]\delta[\rho]\end{aligned} \tag{4.14}$$

with $R^{(0)}_{(n)}\delta[\rho]$ denoting the n-th order ρ-derivative term in the expansion of $W_Y^{(0)}$ in Eq. (3.1). Fig. 4 shows Eq. (4.14) in graphical form. *Note that the dual evolution*

allows for a single target dipole to interact also via three and four gluon exchange while in the dipole model a single target dipole interacts only via two gluon exchange.

The second step of the evolution follows simply by applying the same rule as in the first step for the created dipoles inside the evolved target dipole. The resulting graphs are shown in Fig. 5. As compared with the dipole model, the extra graphs in the dual evolution are those with three and four gluons attached at the end of the BFKL evolution to a single target dipole. Further steps of evolution will only add some more dipoles according to BFKL but will not split the three and four gluons and attach them to different target dipoles. Thus, the rapidity dependence of the evolved target wavefunction where three or four gluons are attached to a single target dipole, $\overline{W}_Y$, is that of a single BFKL Pomeron

$$\overline{W}_{Y\,(i)} \sim \; e^{(\alpha_P-1)Y}\,\overline{W}_{Y=0\,(i)} \tag{4.15}$$

where $i = 3$ or 4 stands for three or four gluons respectively. This leads to the following rapidity dependence of the T-matrix for the scattering of the two projectile dipoles via two-gluon exchange with a single dipole in the evolved target

$$T^{\,1\,\mathrm{Pom}\to 4\mathrm{g}}(Y) \;\sim\; \alpha_s^4\; e^{\,(\alpha_P-1)\,Y}\;. \tag{4.16}$$

In the Pomeron language this result is shown in Fig. 3(c) where a single Pomeron interacts at the end of its evolution via four gluon exchange with two dipoles. Note that the three gluon exchange part $\overline{W}_{Y\,(3)}$ will drop out when considering the interactions with two projectile dipoles.

The wavefunction of an evolved target dipole at the fourth order of ρ-derivatives as given by the dual equation can be separated into two parts in the form

$$W^{\mathrm{dual}}_{Y\,(4)} = W^{\mathrm{MSW}}_{Y\,(4)} + \overline{W}_{Y\,(4)}\,, \tag{4.17}$$

where the first part is in the dipole model and the second is not. Eq. (4.17) tells us that at high rapidities $W^{\mathrm{MSW}}_{Y\,(4)}$ gives the dominant contribution to the scattering matrix since $W^{\mathrm{MSW}}_{Y\,(4)} \gg \overline{W}_{Y\,(4)}$ (see Eq. (4.12) and Eq. (4.15)). This is the same result that was recognized already in Ref. [20]: the double Pomeron interaction (Fig. 3(a) and Eq. (4.13)) dominates over the single Pomeron which interacts at the end of its evolution with the two external dipoles (Fig. 3(c) and Eq. (4.16)).

In the dipole model [24, 25] our main result reflects the fact of dipole counting in the scattering of two external dipoles with the evolved target: when both external dipoles can scatter off different target dipoles, each of them participates with the factor $\alpha_s^2\; n(Y)$, thus, giving $T \sim [\alpha_s^2\; n(Y)]^2$, with the dipole number density $n(Y) \sim \exp\,[(\alpha_P - 1)Y]$. On the other hand, when both external dipoles are forced to scatter

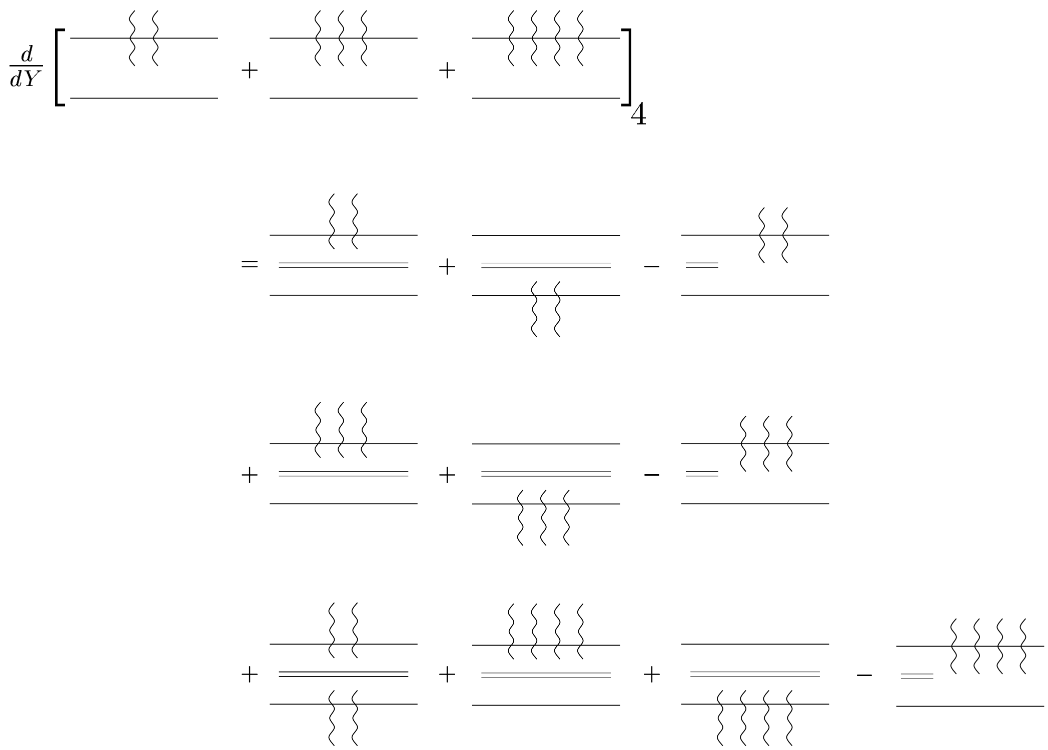

Figure 4: One step of evolution of an unevolved single dipole according to the dual equation up to the fourth order in ρ-derivatives.

off a single target dipole, the one scattering first gives the factor $\alpha_s^2\, n(Y)$ while the second one has only one possibility $\alpha_s^2 \times 1$, leading to $T \sim [\alpha_s^4\, n(Y)]$. Note that only at very low rapidity where $n(Y)$ is of order one, the two cases give similar results.

5 Scheme dependence of the evolution kernel; explicit calculations

In Ref. [15] the evolution in rapidity of the target wave function was written in the Hamiltonian form. Two versions of this Hamiltonian were given explicitly in terms of derivatives of ρ up to the fourth derivative: one for the general case and the other for when one is dealing exclusively with dipoles. The problem is that the derivative form of these two Hamiltonians did not appear to be equivalent to each other. On top of that the dipole version of this Hamiltonian was emphatically pointed out to differ from the MSW term. The reason for the latter was partly

$$\frac{d}{dY}\left[\begin{array}{c}\text{graphs on the}\\ \text{rhs of Fig. 4}\end{array}\right]_4 = \left[\begin{array}{c}\text{graphs from the dipole model}\\ \text{on the rhs of Fig. 2}\end{array}\right]$$

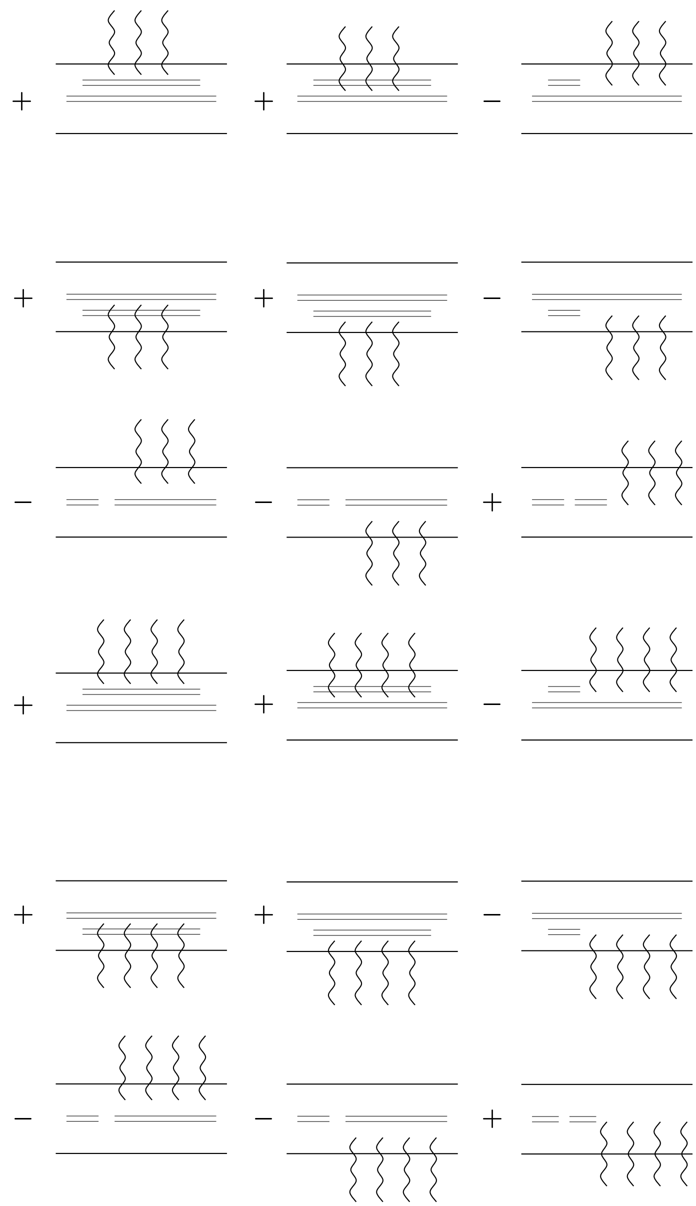

Figure 5: One step of evolution of the graphs on the r.h.s. of Fig. 4 according to the dual evolution up to the fourth order in ρ-derivatives.

explained in Sec. 4. There remains the disagreement of the general derivative form and the dipole derivative form of this Hamiltonian in [15]. It is the purpose of this section to show that one should not compare these Hamiltonians among themselves as well as comparing them with the MSW term directly because the specific form of the Hamiltonian itself is not unique hence the scheme dependence (see below). For this reason it only makes sense to compare physical quantities such as the action of these Hamiltonians on the target wave function. Those are the quantities that should be in agreement independent of the form of the Hamiltonian.

From the discussion in Sec. 3 it is clear that our general form of W_Y, evolved from a parent dipole with transverse coordinate $(\boldsymbol{x},\boldsymbol{y})$, is

$$W_Y(\boldsymbol{x},\boldsymbol{y}) = \sum_{N=1}^{\infty} \int d\Gamma_N P_N(Y) \prod_{i=1}^{N} \int_{\boldsymbol{x}_{i-1},\boldsymbol{x}_i} R(\boldsymbol{x}_{i-1},\boldsymbol{x}_i)\, \delta[\rho] \tag{5.1}$$

where $R(\boldsymbol{x},\boldsymbol{y})$ is as given in Eq. (3.1) while $P_N(Y)$ gives the probability to find a system of N "dipoles" at transverse coordinates $(\boldsymbol{x}_0,\boldsymbol{x}_1)$, $(\boldsymbol{x}_1,\boldsymbol{x}_2)$, $\ldots$ $(\boldsymbol{x}_{N-1},\boldsymbol{x}_N)$ with the first coordinate of the pair $(\boldsymbol{x}_{i-1},\boldsymbol{x}_i)$ denotes a "quark" and the second an "antiquark" with $\boldsymbol{x}_0 = \boldsymbol{x}$ and $\boldsymbol{x}_N = \boldsymbol{y}$, matching onto the parent dipole. One can write the evolution of W_Y as

$$\frac{\partial W_Y}{\partial Y} = \chi\, W_Y\ . \tag{5.2}$$

In Eq. (3.2) we have taken

$$\chi = -\frac{1}{2}\int_{\boldsymbol{x},\boldsymbol{y}} \rho^a(\boldsymbol{x})\, \eta^{ab}(\boldsymbol{x},\boldsymbol{y})[-i\,\delta/\delta\rho]\, \rho^b(\boldsymbol{y}) \tag{5.3}$$

with η given in Eq. (2.9). While this is a natural choice for χ it is not unique. Indeed the choice made in Eq. (2.33) of [15] is not the same as we have taken although it is an equally good choice. The product $\chi\, W_Y$ in Eq. (5.2) is unique, however there are different χ's which acting on W_Y, give the same result.

The ambiguity in χ is a sort of scheme dependence and can be illustrated by considering the action of $\rho^a(u^+,\boldsymbol{u})$ on $U^\dagger_{\boldsymbol{x}}$. It is then straightforward to verify that

$$\rho^a(u^+,\boldsymbol{u})\, U^\dagger_{\boldsymbol{x}}\, \delta[\rho] = \{\tilde{U}_{\boldsymbol{u}}(u'^+,u^+)\}^{ab}\, \rho^b(u'^+,\boldsymbol{u})\, U^\dagger_{\boldsymbol{x}}\, \delta[\rho] \tag{5.4}$$

where

$$\tilde{U}_{\boldsymbol{u}}(u'^+,u^+) = \overline{P}\exp\Big(-g\int_{u^+}^{u'^+} dx^+ T^a \delta/\delta\rho^a(x^+,\boldsymbol{u})\Big) \tag{5.5}$$

and where $\overline{P}$ indicates that larger x^+-components of the integral are put further to the right. However it is not true that

$$\rho^a(u^+,\boldsymbol{u}) = \tilde{U}^{ab}_{\boldsymbol{u}}(u'^+,u^+)\, \rho^b(u'^+,\boldsymbol{u})\ . \tag{5.6}$$

The fact that Eq. (5.6) is not generally correct can easily be checked by considering the commutator of the operator, $\delta^c_{\boldsymbol{x}}$, defined earlier in Eq. (4.2) (we repeat the definition here for convenience)

$$\delta^c_{\boldsymbol{x}} = g \int_{-\infty}^{\infty} dx^+ \frac{\delta}{\delta\rho^c(x^+, \boldsymbol{x})}\,, \tag{5.7}$$

on the left and right hand sides of Eq. (5.6). Clearly

$$[\delta^c_{\boldsymbol{x}}, \rho^a(u^+, \boldsymbol{u})] = g\,\delta_{ac}\,\delta^2(\boldsymbol{x}-\boldsymbol{u}) \tag{5.8}$$

while

$$[\delta^c_{\boldsymbol{x}}, \tilde{U}^{ab}_{\boldsymbol{u}}(u'^+, u^+)\rho^b(u^+, \boldsymbol{u})] = g\,\delta^2(\boldsymbol{x}-\boldsymbol{u})\,\tilde{U}^{ac}_{\boldsymbol{u}}(u'^+, u^+) \tag{5.9}$$

and the right hand sides of Eq. (5.8) and Eq. (5.9) are not the same.

To see this scheme dependence more explicitly we now evaluate χ through terms including four factors of $\delta/\delta\rho$, the same level of accuracy as kept in [9] and in [15]. From Eq. (2.9), Eq. (5.2) and Eq. (5.3)

$$\chi = -\frac{1}{2\pi}\int_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\rho^a(\boldsymbol{x})\,M_{ab}\,\rho^b(\boldsymbol{y}) \tag{5.10}$$

where

$$M_{ab} = \{1 + \tilde{U}^\dagger_{\boldsymbol{x}}\tilde{U}_{\boldsymbol{y}} - \tilde{U}^\dagger_{\boldsymbol{x}}\tilde{U}_{\boldsymbol{z}} - \tilde{U}^\dagger_{\boldsymbol{z}}\tilde{U}_{\boldsymbol{y}}\}^{ab}\,. \tag{5.11}$$

To simplify the discussion a little we suppose that $\chi\, W_Y$ will be used to evaluate the scattering of two separate dipoles on the target. In that case M_{ab} can be expressed in terms of the operators, $\delta^a_{\boldsymbol{x}}$, defined in Eq. (4.2), while in the general case the x^+-dependence of derivatives with respect to ρ must be kept in M_{ab}. Then it is straightforward to get

$$M^{(2)}_{ab} = -\{T^cT^d\}^{ab}\,(\delta^c_{\boldsymbol{x}} - \delta^c_{\boldsymbol{z}})\,(\delta^d_{\boldsymbol{y}} - \delta^d_{\boldsymbol{z}}) \tag{5.12}$$

$$M^{(3)}_{ab} = -\frac{1}{2}\{T^cT^dT^e\}^{ab}\left[\delta^c_{\boldsymbol{x}}(\delta^d_{\boldsymbol{x}} - \delta^d_{\boldsymbol{y}})\delta^e_{\boldsymbol{y}} - \delta^c_{\boldsymbol{x}}(\delta^d_{\boldsymbol{x}} - \delta^d_{\boldsymbol{z}})\delta^e_{\boldsymbol{z}} - \delta^c_{\boldsymbol{z}}(\delta^d_{\boldsymbol{z}} - \delta^d_{\boldsymbol{y}})\delta^e_{\boldsymbol{y}}\right] \tag{5.13}$$

$$\begin{aligned} M^{(4)}_{ab} = -\frac{1}{12}\{T^cT^dT^eT^f\}^{ab}\Big[&\delta^c_{\boldsymbol{z}}\delta^d_{\boldsymbol{z}}\delta^e_{\boldsymbol{z}}\delta^f_{\boldsymbol{z}} - 2\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{z}}\delta^e_{\boldsymbol{z}}\delta^f_{\boldsymbol{z}} - 2\delta^c_{\boldsymbol{z}}\delta^d_{\boldsymbol{z}}\delta^e_{\boldsymbol{z}}\delta^f_{\boldsymbol{y}} + 3\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{x}}\delta^e_{\boldsymbol{z}}\delta^f_{\boldsymbol{z}} \\ &+ 3\delta^c_{\boldsymbol{z}}\delta^d_{\boldsymbol{z}}\delta^e_{\boldsymbol{y}}\delta^f_{\boldsymbol{y}} - 2\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{x}}\delta^e_{\boldsymbol{x}}\delta^f_{\boldsymbol{z}} - 2\delta^c_{\boldsymbol{z}}\delta^d_{\boldsymbol{y}}\delta^e_{\boldsymbol{y}}\delta^f_{\boldsymbol{y}} + 2\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{x}}\delta^e_{\boldsymbol{x}}\delta^f_{\boldsymbol{y}} \\ &+ 2\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{y}}\delta^e_{\boldsymbol{y}}\delta^f_{\boldsymbol{y}} - 3\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{x}}\delta^e_{\boldsymbol{y}}\delta^f_{\boldsymbol{y}}\Big]\,. \end{aligned} \tag{5.14}$$

A casual glance at Eq. (5.14) reveals that this is not the same expression as that given by Eq. (2.38) in [15]. In particular Eq. (5.14) has terms which have no $\delta_{\boldsymbol{z}}$

factors while the formula in [15] has at least one factor of $\delta_{\boldsymbol{z}}$ in each of its terms. As we have no reason to expect agreement when comparing comparable terms in χ itself, the lack of an agreement is not surprising.

We now turn to the product $\chi\, W_Y$ which is not scheme dependent, and we shall see how the factors in Eq. (5.12), Eq. (5.13) and Eq. (5.14) organize thermselves into a simple expression when the appropriate factors in W_Y are included. To make the calculation easier we take W_Y as

$$W_Y^{(0)} = R(\boldsymbol{x},\boldsymbol{y})\,\delta[\rho]\ . \tag{5.15}$$

Because of the structure of Eq. (5.1), this is no essential limitation. Then letting $\mathcal{K}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \to -\frac{1}{2}\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$,

$$\chi\, W_Y^{(0)} = -\frac{g^2}{2\pi N_c}\int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\, M_{ab} N_{ab}\,\delta[\rho] \tag{5.16}$$

and

$$N_{ab} = \mathrm{tr}(t^a U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}} t^b)\ . \tag{5.17}$$

It is straightforward to evaluate N_{ab} at the lowest few orders in the number of ρ-derivatives. The results are

$$N_{ab}^{(0)} = \frac{1}{2}\delta_{ab} \tag{5.18}$$

$$N_{ab}^{(1)} = \mathrm{tr}(t^b t^a t^c)(\delta^c_{\boldsymbol{x}} - \delta^c_{\boldsymbol{y}}) \tag{5.19}$$

$$N_{ab}^{(2)} = \frac{1}{2}\mathrm{tr}(t^b t^a t^c t^d)\,(\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{x}} + \delta^c_{\boldsymbol{y}}\delta^d_{\boldsymbol{y}} - 2\delta^c_{\boldsymbol{x}}\delta^d_{\boldsymbol{y}}) \tag{5.20}$$

where, as for M_{ab}, the superscript denotes the order of the derivatives in ρ. In the product of $M_{ab}N_{ab}$ appearing in Eq. (5.16) the term $M^{(2)}N^{(0)}$ gives the usual BFKL evolution

$$M_{ab}^{(2)} N^{ab\ (0)} \ = \ -\frac{N_c}{2}(\delta^c_{\boldsymbol{x}} - \delta^c_{\boldsymbol{z}})(\delta^c_{\boldsymbol{y}} - \delta^c_{\boldsymbol{z}})\,, \tag{5.21}$$

$M_{ab}^{(3)} N^{ab\ (0)}$ vanishes and $M^{(2)}N^{(1)}$ gives the BFKL evolution of the odderon

$$M_{ab}^{(2)} N^{ab\ (1)} \ = \ -\frac{N_c}{8}\, d^{abc}\,\Big(\delta^a_{\boldsymbol{x}}\delta^b_{\boldsymbol{y}}(\delta^c_{\boldsymbol{x}} - \delta^c_{\boldsymbol{y}}) - \delta^a_{\boldsymbol{x}}\delta^b_{\boldsymbol{z}}(\delta^c_{\boldsymbol{x}} - \delta^c_{\boldsymbol{z}}) - \delta^a_{\boldsymbol{z}}\delta^b_{\boldsymbol{y}}(\delta^c_{\boldsymbol{z}} - \delta^c_{\boldsymbol{y}})\Big) \tag{5.22}$$

while the combination

$$M_{ab}^{(4)} N_{ab}^{(0)} + M_{ab}^{(3)} N_{ab}^{(1)} + M_{ab}^{(2)} N_{ab}^{(2)} \tag{5.23}$$

gives the four derivative terms which are our main concern here.

Using, at large N_c,

$$\begin{aligned}
\mathrm{tr}(t^b t^a t^c t^d) &= \frac{1}{4N_c}(\delta_{ab}\delta_{cd} + \delta_{bd}\delta_{ac}) && (5.24)\\
(T^e T^f)_{ab}\mathrm{tr}(t^b t^a t^c t^d) &= \frac{1}{4}\delta_{cd}\delta_{ef} + \frac{1}{8}\delta_{ce}\delta_{df} && (5.25)\\
\mathrm{tr}(T^c T^d T^e T^f) &= \delta_{cd}\delta_{ef} + \delta_{cf}\delta_{de} + \frac{1}{2}\delta_{ce}\delta_{df} && (5.26)\\
(T^c T^d T^e)_{ab}\mathrm{tr}(t^b t^a t^f) &= -\frac{1}{4}\mathrm{tr}(T^c T^d T^e T^f) && (5.27)
\end{aligned}$$

we find

$$\begin{aligned}
M_{ab}^{(4)} N^{ab\ (0)} = &-\frac{5}{48}(\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}})^2 + \frac{5}{24}\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}}(\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{z}}.\delta_{\boldsymbol{y}}) - \frac{1}{8}\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}})\\
&- \frac{3}{16}((\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}})^2 + (\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})^2) + \frac{5}{24}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})\\
&- \frac{5}{24}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}) + \frac{1}{8}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}} + \frac{3}{16}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}})^2\ , \qquad (5.28)
\end{aligned}$$

$$\begin{aligned}
M_{ab}^{(3)} N^{ab\ (1)} = &\ \frac{1}{8}\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}} - 2\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}) + \frac{1}{8}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}})\\
&+ \frac{3}{16}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} - \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})^2 - \frac{5}{16}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})\\
&+ \frac{3}{16}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}}) + \frac{5}{16}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}})\\
&- \frac{1}{4}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}} - \frac{3}{8}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}})^2\ , \qquad (5.29)
\end{aligned}$$

and

$$\begin{aligned}
M_{ab}^{(2)} N^{ab\ (2)} = &-\frac{1}{8}\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}} - 2\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}) - \frac{1}{16}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} - \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})^2\\
&+ \frac{3}{16}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}}) - \frac{3}{16}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})\\
&- \frac{3}{16}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}) + \frac{1}{8}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}} + \frac{1}{4}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}})^2\ . \qquad (5.30)
\end{aligned}$$

Putting everything together, the result is

$$\begin{aligned}
&M_{ab}^{(4)} N^{ab\ (0)} + M_{ab}^{(3)} N^{ab\ (1)} + M_{ab}^{(2)} N^{ab\ (2)}\\
&= \frac{1}{48}\Big\{10\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}}(\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{z}}.\delta_{\boldsymbol{y}}) - 5(\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}})^2 - 6\delta_{\boldsymbol{z}}.\delta_{\boldsymbol{z}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}})\\
&\quad - 3((\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}})^2 + (\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})^2 - (\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}})^2) + 4(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}})\\
&\quad + 6(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}}\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}}) - 12\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{z}}\delta_{\boldsymbol{y}}.\delta_{\boldsymbol{z}} - 4\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}}(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}}.\delta_{\boldsymbol{y}})\Big\}
\end{aligned}
\qquad (5.31)$$

Referring back to Eq. (5.16) one now finds the equality

$$\chi\, W_Y^{(0)} = -\frac{g^2 N_c}{2\pi} \int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \Big[R(\boldsymbol{x},\boldsymbol{y}) - R(\boldsymbol{x},\boldsymbol{z}) R(\boldsymbol{z},\boldsymbol{y}) \Big]_{(4)} \delta[\rho] \tag{5.32}$$

where the subscript (4) in Eq. (5.32) indicates that the term in brackets is to be evaluated to fourth order in ρ-derivatives. We note for completeness that

$$R^{(0)}_{(2)}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4N_c}(\delta_{\boldsymbol{x}} - \delta_{\boldsymbol{y}})^2 \tag{5.33}$$

$$R^{(0)}_{(3)}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{6N_c}\,\mathrm{tr}(t^a t^b t^c)(\delta^a_{\boldsymbol{x}}\delta^b_{\boldsymbol{x}}\delta^c_{\boldsymbol{x}} - 3\delta^a_{\boldsymbol{x}}\delta^b_{\boldsymbol{x}}\delta^c_{\boldsymbol{y}} + 3\delta^a_{\boldsymbol{x}}\delta^b_{\boldsymbol{y}}\delta^c_{\boldsymbol{y}} - \delta^a_{\boldsymbol{y}}\delta^b_{\boldsymbol{y}}\delta^c_{\boldsymbol{y}}) \tag{5.34}$$

and

$$R^{(0)}_{(4)}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{96N_c^2}[3(\delta_{\boldsymbol{x}}-\delta_{\boldsymbol{y}})^2(\delta_{\boldsymbol{x}}-\delta_{\boldsymbol{y}})^2 - \delta^2_{\boldsymbol{x}}\delta^2_{\boldsymbol{x}} - \delta^2_{\boldsymbol{y}}\delta^2_{\boldsymbol{y}} - 6(\delta_{\boldsymbol{x}}.\delta_{\boldsymbol{y}})^2 + 4\delta_{\boldsymbol{x}}\cdot\delta_{\boldsymbol{y}}(\delta^2_{\boldsymbol{x}}+\delta^2_{\boldsymbol{y}})]\ . \tag{5.35}$$

It is now clear that the fourth order terms in Eq. (5.16) agrees with the general expression Eq. (3.15) derived earlier. It is also clear that the simple and elegant form embodied in Eq. (3.15) and Eq. (5.32) is not embodied in χ alone but is rather a property of the product $\chi\, W_Y$.

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